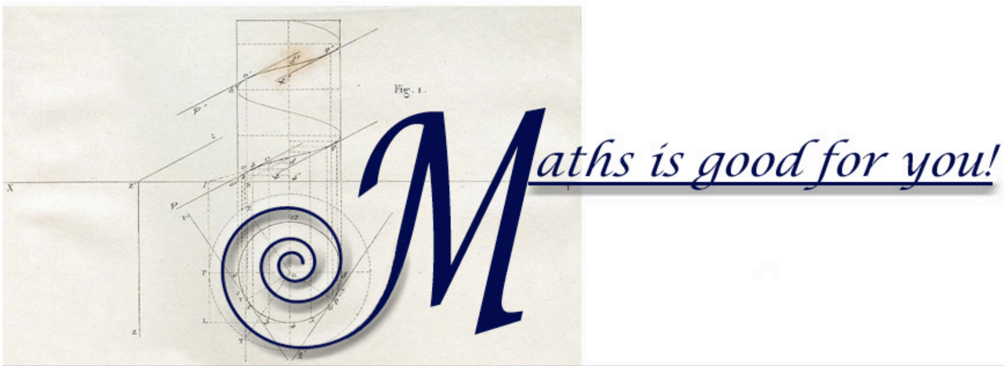

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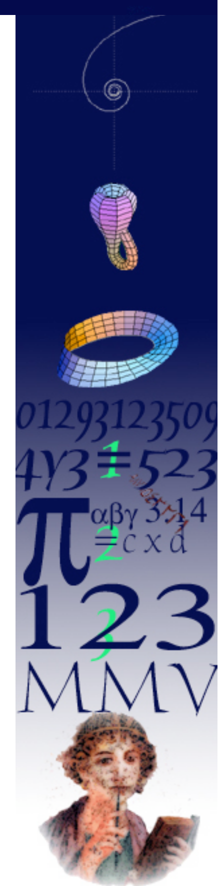

Snezana Lawrence

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CONTENTS

SURDS AND INDICES

INTRODUCTION – NUMBER SETS

Real numbers ($\mathbb{R}$) are all numbers that correspond to some number on a number line (whether whole – integers, or decimal numbers, positive or negative).

The set of real numbers can be separated into two subsets: rational ($\mathbb{Q}$) and irrational (no sign for them) numbers. All rational numbers can be expressed as fractions of two integers, whereas the irrational numbers cannot.

Natural numbers are positive counting numbers (1, 2, 3, etc.). They are denoted $\mathbb{N}$. Integers are whole numbers, whether positive or negative, denoted $\mathbb{Z}$ (and there are two subsets to this – positive integers, $\mathbb{Z}^+$, and negative, $\mathbb{Z}^-$).

Roots that cannot be expressed as rational numbers are called SURDS. Surds are expressions containing an irrational number – they cannot be expressed as fractions of two integers. Examples are $\sqrt{2}, \sqrt{11}, \sqrt{10}$. When asked to give an exact answer, leave the square roots in your answer (although you may have to manipulate and/or rationalise the denominator) as you can never give an exact value of an irrational number.

RATIONALISING THE DENOMINATOR

For historical reasons, you need to somehow always 'get rid' of the surd if it is in the denominator. There are several techniques of manipulation you can use.

When multiplying or dividing with surds, this may be helpful

$$\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$$

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

$$\sqrt{a} \times \sqrt{a} = \sqrt{a^2} = a$$

When you get the expression which does not only have a surd in the denominator, but something like this:

$$\frac{1}{\sqrt{5}-2}$$

you need to think how to get the surd in the numerator and have denominator without any surds. So you may want to apply the difference of squares:

$$(x-y)(x+y) = x^2 - y^2$$

This can also be applied to the case when one, or both, of the members of the brackets are square roots (or surds):

$$\left(\sqrt{x}-\sqrt{y}\right)\left(\sqrt{x}+\sqrt{y}\right) = x - y$$

So

$$\frac{1}{\sqrt{5}-2}=\frac{1}{\sqrt{5}-2}\bullet\frac{\sqrt{5}+2}{\sqrt{5}+2}=\frac{\sqrt{5}+2}{\left(\sqrt{5}\right)^2-2^2}=\frac{\sqrt{5}+2}{5-4}=\sqrt{5}+2$$

LAWS ON INDICES

To be good with surds, you also need to know all rules for **INDICES**:

$$a^0=1$$

$$a^{-1}=\frac{1}{a},and\left(\frac{a}{b}\right)^{-1}=\left(\frac{b}{a}\right)$$

$$a^n\times a^m=a^{n+m}$$

$$a^n\div a^m=\frac{a^n}{a^m}=a^{n-m}$$

$$a^{\frac{n}{m}}=\sqrt[m]{a^n}$$

$$a^m\times b^m=\left(ab\right)^m,and\left(\frac{a}{b}\right)^m$$

COMMON MISTAKE

Never, ever split the sum or difference of an expression like $\sqrt{x+y}$ into $\sqrt{x}+\sqrt{y}$. You can't do that because you have two different numbers in an expression which is of the form $(x+y)^{\frac{1}{2}}$. And as with any other powers of expressions within the brackets you can't just apply the powers to the members of the brackets, but you must imagine these bracketed expressions multiplying each other as many times as the power says.

What you CAN do is break a number or a product under a root into its factors, some of which will hopefully be numbers that you will be able to take outside of the root. For example,

$$\sqrt{27}=\sqrt{9\bullet3}=3\sqrt{3}$$

or

$$\sqrt{8x^3}=\sqrt{4\bullet2\bullet x^2\bullet x}=2x\sqrt{2x}$$

ALGEBRA – DIFFERENCE OF SQUARES AND FEW OTHER THINGS

ALGEBRAIC EXPRESSIONS

A term consists of products of numbers and letters; for example $3a^2 + 4b$ is a **term**. The numbers multiplying the letters in an expression or equation are called **coefficients**; for example, *a* and *b* in the following expression ax^2+bx are coefficients.

FUNCTIONS AND POLYNOMIALS

Relationship between an input value and the output value defined by mathematical expression is called function. Function can be written as
$y = x^2$, or $f(x) = x^2$. In both cases, this statement says that for any value of *x*, the value of *y*, or *f(x)* will be the square of the original (input) value.

Linear function has a form $f(x) = mx + c$, and the corresponding graphical presentation of this function will be a straight line.

Quadratic function has a form $f(x) = ax^2 + bx + c$, and the corresponding graphical presentation will be a parabola.

Polynomial has a form $a_n x^n + a_{n-1} x^n + ... + a_0 x^0$. The degree of polynomial is determined by the greatest power of *x*, so in this example, polynomial would have n^{th}-degree.

USEFUL PRODUCTS AND FACTORISATIONS FOR ALGEBRAIC MANIPULATION

These are to be memorised if you can – they can be very useful to you.

$$(a+b)(a-b) = a^2 - b^2$$
$$(x+a)(x+b) = x^2 + (a+b)x + ab$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$
$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

COMMON MISTAKE

Don't ever do something like $(a+b)^2 = a^2 + b^2$ because you have missed the point that $(a+b)^2$ actually means $(a+b)(a+b)$, which will give you, as above,
$(a+b)^2 = a^2 + 2ab + b^2$.

SOLVING QUADRATIC EQUATIONS

GENERALLY SPEAKING…

There are three main ways to solve a quadratic equation:

1. By trying to factorise it; so that you get an equation of the sort

$$(x-a)(x-b)=0$$

then it is easily seen that if $(x-a)(x-b)=0$ either $x=a$, or $x=b$

2. By using the formula for solving the quadratic equation, which relates to

$$ax^2+bx+c=0$$

and the two solutions would be $x_{1,2}=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

3. By completing the square, so that you get something like

$$(x-3)^2-25=0$$

which means that you can write that as

$$(x-3)^2=25$$

and then you can take square root of both sides, which gives you

$$(x-3)=\pm\sqrt{25}$$

so, $x=3\pm 5$.

COMPLETING THE SQUARE

Always look to have coefficient with *x* to be *a*=1, or manipulate your equation until you have it like that.

Let us say you have a usual quadratic of a form

$$ax^2+bx+c=0\text{, where } a=1$$

You need to use half of *b*, so let us introduce a value *k* such that $k=\dfrac{b}{2}$.

Re-write your equation as

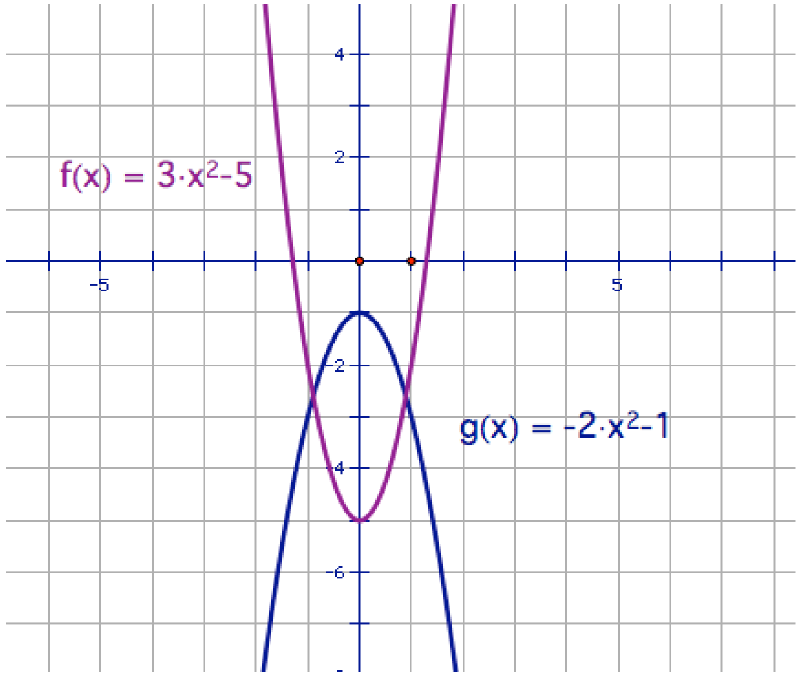
$$(x+k)^2-k^2+c=0$$

Because neither k^2 nor *c* have *x* 'attached' to them, you can tidy the equation up and that will be completed square of the original equation.

You can also use the completed square form to sketch the graph of a quadratic faster. The information given by the completed square form is

$$y=a(x+p)^2+q$$

The vertex of parabola will have coordinates (-*p*,*q*); its axis of symmetry will be *x*=-*p*. It will be ∪ shaped if $a > 0$, and ∩ shaped if $a < 0$.



SKETCHING QUADRATICS

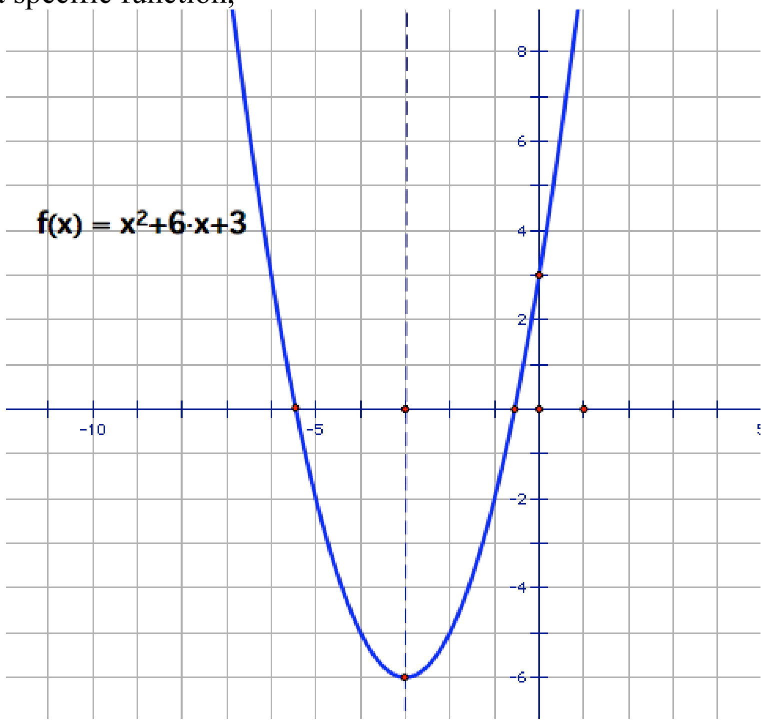
FIRST THINGS FIRST - WHAT IS QUADRATIC?

Function of a form

$y = ax^2 + bx + c$

is called quadratic – and its graph is a parabola.

Let us look at a specific function,



To sketch it, follow these steps:

1. First look at the coefficient next to x^2 (remember that would be a). There are two possibilities
   - $a < 0$ in which case the parabola would be side up – like a hill
   - $a > 0$ in which case the parabola is like a valley

   In our case, $a > 0$, so it is a 'valley'.

2. You can now look at the formula for finding the roots of the function:

   $$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

   The part under the square root is called the discriminant; there are three possibilities
   - $\sqrt{b^2 - 4ac}$ >0, in which case the function has two roots (the parabola cuts the x-axis in two places
   - $\sqrt{b^2 - 4ac}$ =0, in which cases there is a repeated root, the parabola only touches the x-axis

- $\sqrt{b^2 - 4ac}$ <0, in which case there are no real roots – the parabola will not cut or touch the x-axis.

3. Now complete the square for this function (equate it with 0, to find the roots at the same time). For the given function, $x^2 + 6x + 3 = 0$, the completed square form would be $(x+3)^2 - 9 + 3 = 0$, or when tided up, $(x+3)^2 - 6 = 0$.

   The roots are $x = -3 \pm \sqrt{6}$

4. The completed square form gives us various information; let us think of it as being generally of a form $a(x+p)^2 + q$

   a. The axis of symmetry for the parabola will be = -p ; in our case –3
   b. The vertex of this parabola will be (-p, q); in our case (-3,-6)

5. We could have found the roots by either using 2, or 3, as above. In either case, we have two distinct roots in our example, and they are $-3 \pm \sqrt{6}$. These are two points at which parabola cuts the x axis, (0, $-3+\sqrt{6}$), and (0, $-3-\sqrt{6}$).

6. To find where the parabola intersects with the y-axis, substitute $x = 0$
   In our case, that would be at $y = 3$.

INEQUALITIES

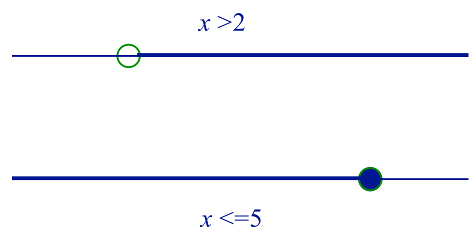
LINEAR INEQUALITIES

When dealing with inequalities solve them as equations, but maintain the inequality sign as it originally held. It is unsafe to square both sides or take their reciprocals with the inequality in place, so be careful when doing that!

If you have two or more inequalities and you are trying to find the range of values for the unknown for which they will hold, the easiest thing to do is to draw a number line and find where the values hold in all cases. For example, find the range of values for which $5<2x+1\leq 11$; solving the two separately would give us

$$5 < 2x+1 \Leftrightarrow 4 < 2x \Leftrightarrow x > 2$$
$$2x+1 \leq 11 \Leftrightarrow 2x \leq 10 \Leftrightarrow x \leq 5$$

and on the number lines that would like this



which means that the values for which both inequalities hold are $2 < x \leq 5$

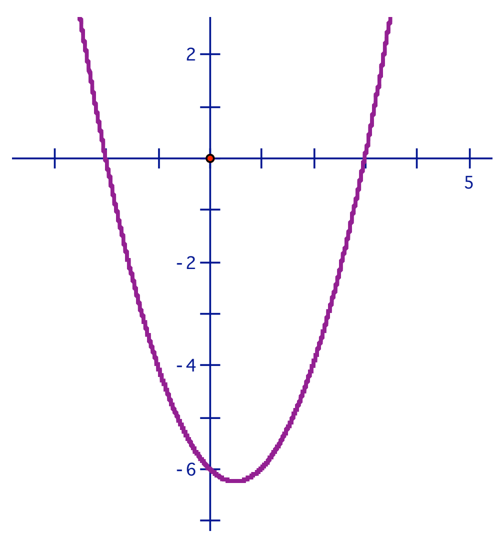
QUADRATIC INEQUALITIES

The easiest way is to find the roots and factorise the quadratic inequality. Look at the signs to decide on the range of values of *x* required to make the inequality true.

For example, solve $x^2 - x - 6 < 0$

Factorise the corresponding equation $(x-3)(x+2)=0$
Which means that either is $x=3$, or $x=-2$.

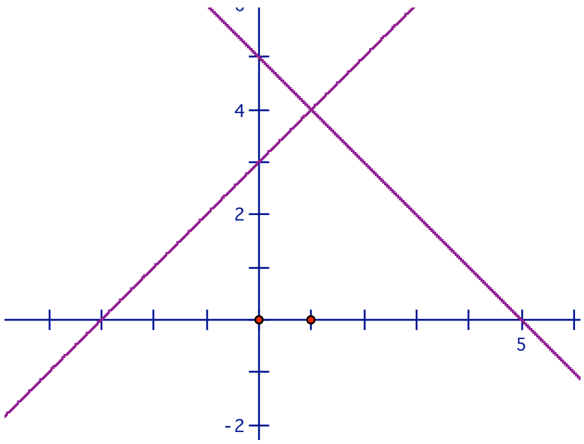
If we sketch the graph we would see that the range of values for which *y*<0 is enclosed by the parabola and the *x* axis. Therefore the solution to the original inequality is



$$-2 < x < 3$$

SIMULTANEOUS EQUATIONS

DIFFERENT CASES



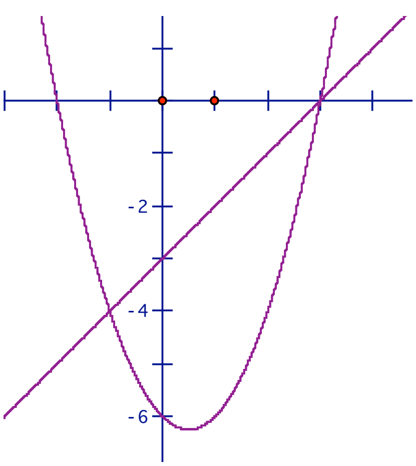
Two linear equations – their single point of intersection will result in one value for x and one value for y which will be the solutions to these two simultaneous equations.

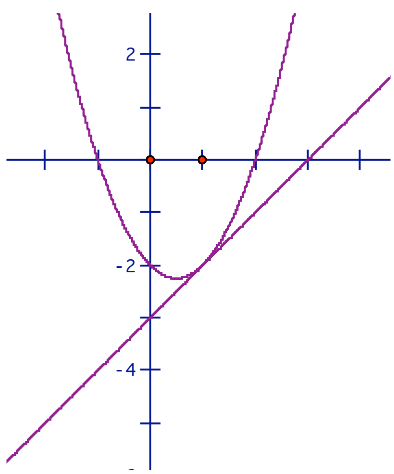
If you have a quadratic and linear equations there are three possible cases:

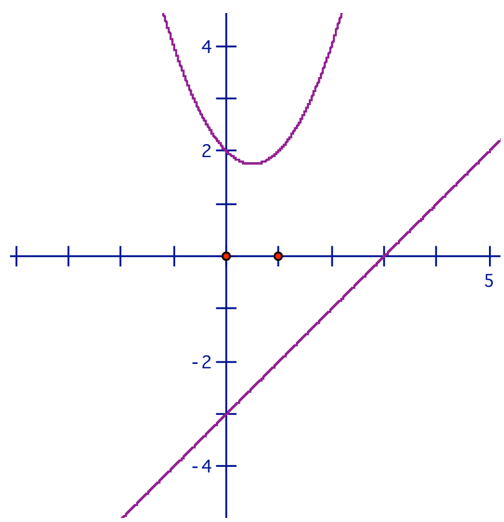
There will be two points of intersection, therefore 2 values x and 2 for y.

There is one point of intersection, 1 x and 1 y.

No intersections between the two lines, therefore no common solutions.







USE OF THE DISCRIMINANT

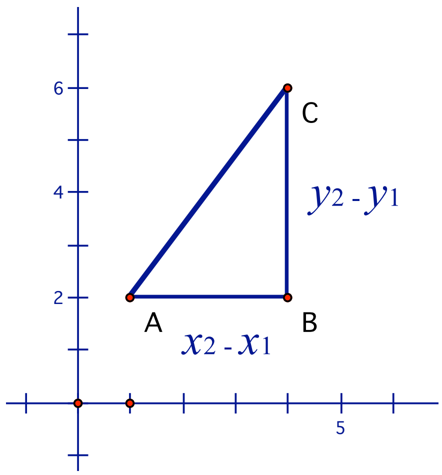
Without drawing the two graphs, you can use algebraic manipulation to find the same: rearrange the linear equation and substitute it into the quadratic. A new quadratic is formed; use the discriminant of this new equation to see how many roots (common solutions) you will have for the two original equations.

If $d > 0$ – there are two distinct roots
$d = 0$ – one real root (repeated root)
$d < 0$ – no real roots.

COORDINATE GEOMETRY OF STRAIGHT LINE

LINE SEGMENTS

The length of the line segment joining two points will relate to their coordinates. Have a good look at the diagram

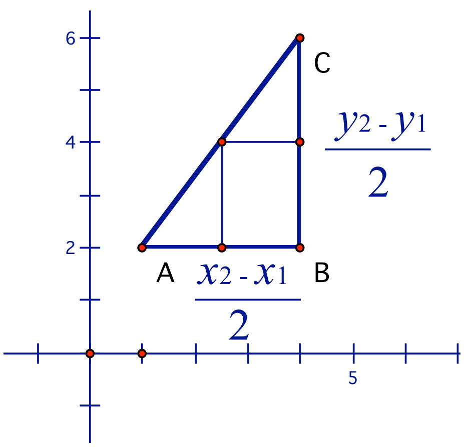


The length joining the point A and C can be found by using Pythagoras' Theorem:

$$AB^2 + BC^2 = AC^2$$

$$AC = \sqrt{AB^2 + BC^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Mid-point of the line can be found by using the same principle



So the point between A and C will have the coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

THE GRADIENT OF A LINE

The gradient measures the steepness of the line.

It is defined as $\dfrac{increase'y'}{increase'x'}$, or $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

When the gradient is 1, the line makes a 45^0 angle with either axes. If the gradient is 0, the line is parallel to the x axis.

Parallel and Perpendicular Lines

When two lines are parallel, their gradient is the same: $m_1 = m_2$

When two lines are perpendicular, their product equals -1: $m_1 m_2 = -1$.

Equation of a line

Forms of equations of straight line

$y = mx + c$	Line gradient *m*, the intercept on *y*-axis is c
$y = mx$	Line gradient *m*, passes through the origin
$y = x + c$	Line gradient 1, makes an angle of 45^0 with the *x*-axis, and the intercept on the *y* axis is *c*
$y = k$	Line is parallel to the *x* axis, through (0, *k)*
$y = 0$	Line is the *x* axis
$x = k$	Line is parallel to the *y* axis, through (*k*, 0)
$x = 0$	Line is the *y* axis
$ax + bx + c = 0$	General form of the equation of a straight line

Common tasks

… are of two types – find the equation of a line, when you have:

1. **one point and a gradient**

   Let the point be given by the coordinates (x_1, y_1)

   The gradient is given as *m*.

   Then substitute the given values into

   $$y - y_1 = m(x - x_1)$$

2. **two points**

   Let the given points be those with the coordinates (x_1, y_1) and (x_2, y_2).

   Then you can find the gradient *m* by substituting these values into

   $$m = \frac{y_2 - y_1}{x_2 - x_1}$$

   You can then use the formula $y - y_1 = m(x - x_1)$ and substitute the given values to get the equation.

To see whether a point lies on a line (whether straight or a curve), substitute the values of its coordinates into the equation. If the point indeed lies on that line, the equation will be satisfied, if not you will get an incorrect equation.

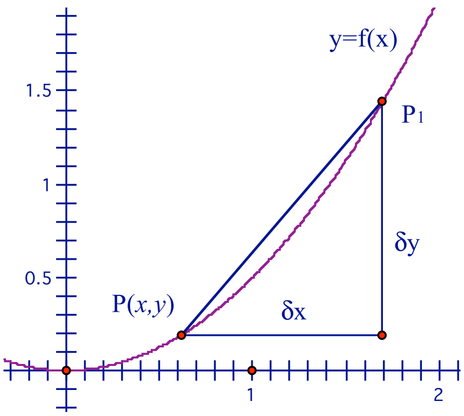
DIFFERENTIATION

RATES OF CHANGE

The gradient of a straight line measures how steep the line is. It is calculated from the difference in y over the difference in x.

Tangent to a curve at any point is the straight line which touches the curve at that point. It also gives a gradient of the curve at that point.

The process of finding the gradient of a function is called **differentiation**. This is part of what is called '**differential calculus**'.



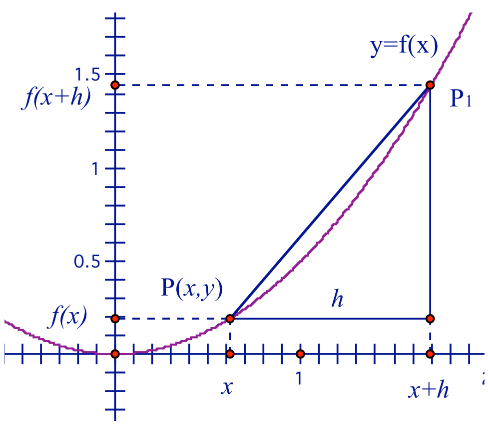
The gradient of PP_1 will be

$PP_1 = \dfrac{\delta y}{\delta x}$, and that means that the gradient at P will be

$$P = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$$

The gradient of a function at a particular point can then be found in this way. It gives a 'derivative' function, and is called:

- The first derivative with respect to x , or
- The first differential coefficient with respect to x, or
- The (first) derived function…
- The gradient function.



In function notation, the gradient is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The first derivative is written as $\dfrac{dy}{dx}$, or y', or f'(x).

The second derivative is written as $\dfrac{d^2y}{dx^2}$, y'', f''(x).

DIFFERENTIATION OF POLYNOMIALS

$$y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}, n \in \mathbb{R}$$

$$y = kx^n \Rightarrow \frac{dy}{dx} = knx^{n-1}$$

$$y = kx \Rightarrow \frac{dy}{dx} = k$$

$$y = c \Rightarrow \frac{dy}{dx} = 0$$

$$y = f(x) \pm g(x) \Rightarrow \frac{dy}{dx} = f'(x) \pm g'(x)$$

$$y = kf(x) \Rightarrow \frac{dy}{dx} = kf'(x)$$

GENERAL RULES OF DIFFERENTIATION

$(cf)' = cf'$

$(f+g)' = f' + g'$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

DERIVATIVES OF SIMPLE FUNCTIONS

Function *f(x)*	**Derivative *f'(x)***
C	0
X	1
Cx	c
x^n	nx^{n-1}
e^x	e^x
ce^{cx}	ce^{cx}
a^x	$a^x \ln a$
$\ln x$	$\frac{1}{x}$
$\log x$	$\frac{1}{x\ln(10)}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\arcsin x = \sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x = \cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan x$	$\sec^2 x = \frac{1}{\cos^2 x}$
$\arctan x = \tan^{-1} x$	$\frac{1}{1+x^2}$

TRANSFORMATION OF GRAPHS AND APPLICATION OF DIFFERENTIATION

APPLICATION OF DIFFERENTIATION – SKETCHING OF FUNCTIONS

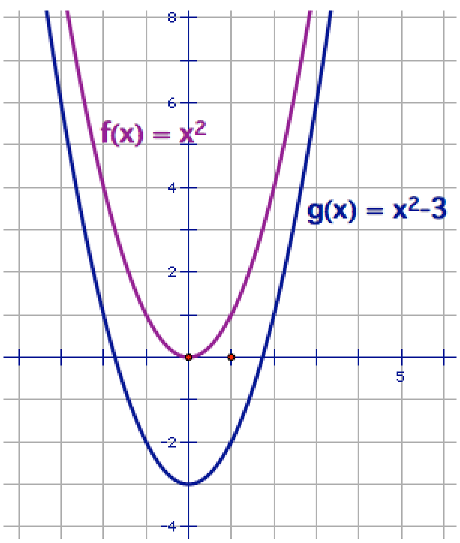
The first derivative of a function gives a gradient of a function at a particular point. If the first derivative is = 0, the function has a stationery point at that point.

To find whether the stationary point is a maximum or a minimum, you need to find the second derivative. These are the rules to remember:

When y'=0, and y''<0, the point is a minimum point
When y'=0, and y''>0, the point is a maximum point
When y'=0, and y''=0, the point is one of inflexion or maximum or minimum.

TRANSFORMATION OF GRAPHS

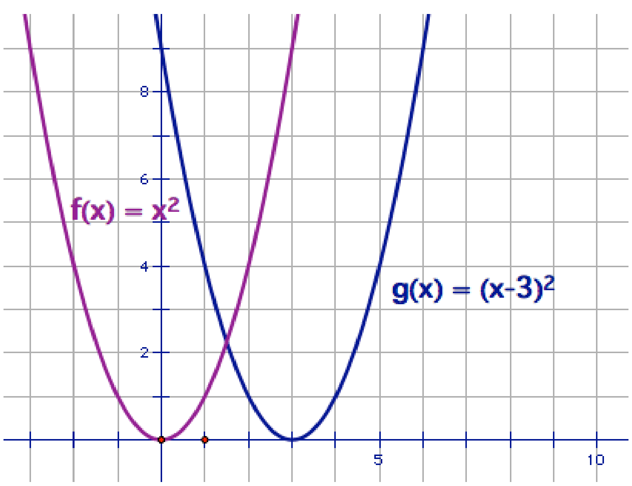
1. Translation along a ***y* axis** is a graph of the function



y = f(x)+a

In this case *a* = -3

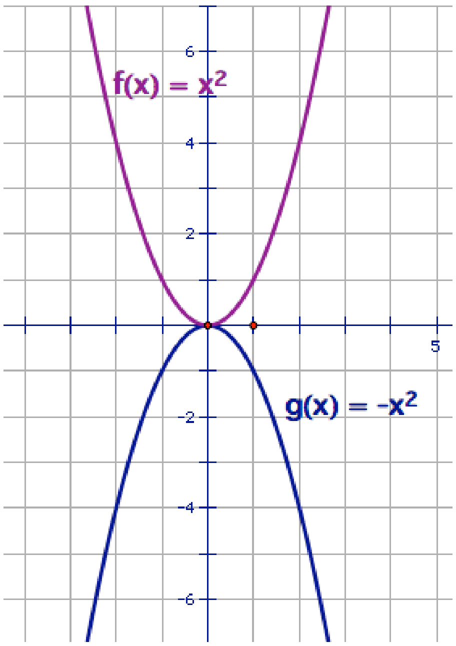
2. Translation along the ***x* axis** is a graph of the function



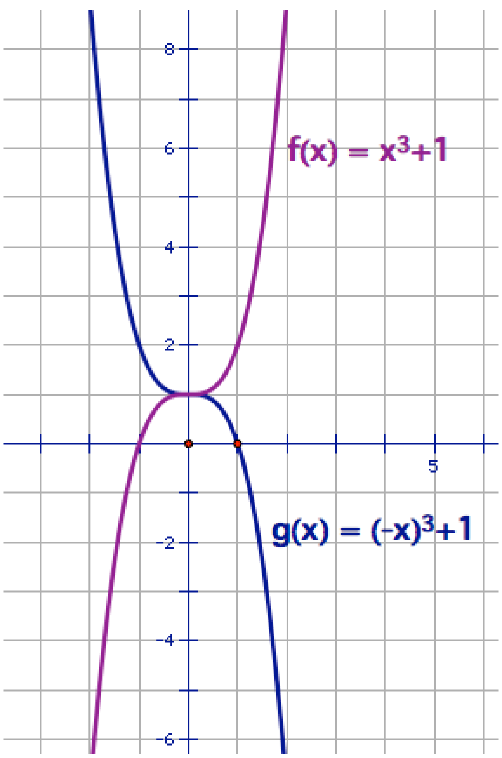
y = f(x+a)

In this case *a* = -3

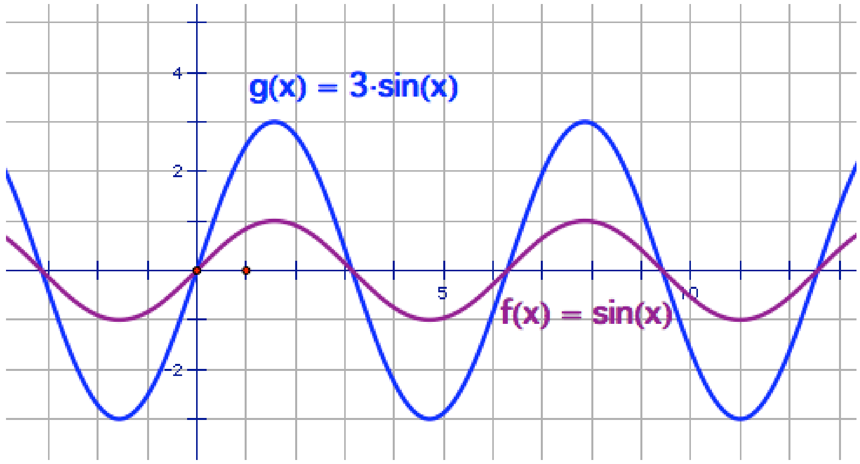
3. The reflection in the *x* axis is a function of the form $y = -f(x)$



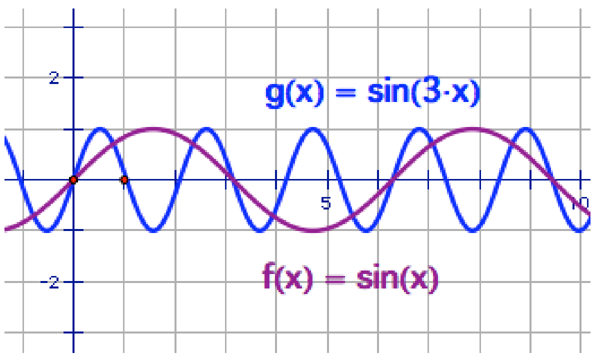
4. The reflection in the *y* axis is a function of the form $y = f(-x)$



5. The 'stretch' in the *y* axis will happen if the function is of a form $y = a\,f(x)$



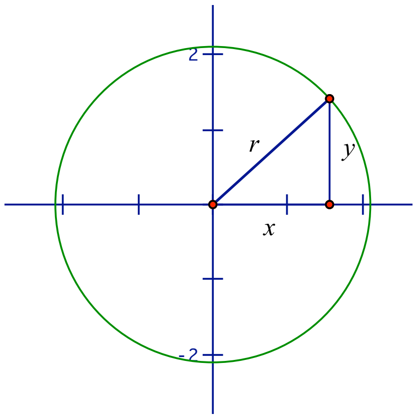
6. The 'stretch' in the *x* axis will happen if the function is of the form $y = f(ax)$



Coordinate Geometry of the Circle

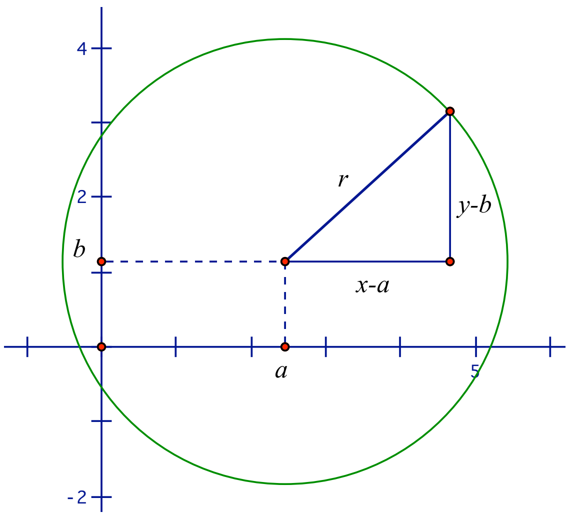
Circle Equation

A circle is a loci of the points equidistant from its centre. It can be defined by its radius and its centre, or by any three points on its circumference.



Looking at the circle drawn at the origin, and simply using Pythagoras' Theorem, you can see that the radius can be easily found once you have x and y coordinates of any of the points on this circle. But more importantly these three values, x, y, and r, you can write the equation of the circle:

$$x^2 + y^2 = r^2$$



If the circle's centre is away from the origin, then you need to take into account the coordinates of the centre. They are on this picture shown as a and b.

The equation of the circle here will be

$$(x-a)^2 + (y-b)^2 = r^2$$

Equation of a Circle and Completing the Square

If you are given an equation of the circle that looks something like this $x^2 + y^2 + 2x - 8y + 8 = 0$, then you need to complete the square for both x and y. In this example, this will give you $(x+1)^2 + (y-4)^2 = 3^2$, which means that the circle whose equation this is has a radius $r = 3$, and the centre at a point (-1, 4).

Problems

To find the equation of the tangent to a circle at a point on its circumference, use the fact that the tangent is perpendicular to the radius.

To find the equation of the normal to a circle at a point on its circumference use the fact that the normal is perpendicular to the tangent. See Coordinate Geometry of a Line.